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Multiple solutions for semilinear resonant elliptic problems with discontinuous nonlinearities via nonsmooth double linking theorem

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Abstract In the present paper, some multiplicity results for semilinear resonant elliptic problems with discontinuous nonlinearities are obtained by using our extended double linking theorem.

Keywords Double linking · Critical point · Locally Lipschitz · Nonsmooth

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1 Introduction

Recently, the nonsmooth analysis based on the critical point theorem has been attracted the interest by many people, because its intensive applications in the practical problems. However, the tools in our hand are used only the usual critical point theorem such as, nonsmooth mountain pass theorem, nonsmooth saddle point theorem (see [1,6,8]). One of the main difficulties in applying those nonsmooth critical point theorems is the nonsmooth Palais Smale condition. Thus, in order to kill this difficulty, we have to take our energy to verify it. In the present paper, our abstract results will not need the energy function to satisfy the PS condition, and we obtain two pairs of bounded Palais Smale sequences. The most interest is that when the energy function satisfies the appropriate conditions, we obtain two solutions.

The notion of double linking was first introduced by Schechter and Zou [14], and a double linking theorem had been developed by them. The purpose of this paper is to present a generalization of the double linking theorem. In this generalization the energy functional is not required to be smooth, it is only locally Lipschitz. In the second half of the paper the abstract multiplicity result is applied to semilinear resonant elliptic problems with discontinuous nonlinearities. Such nonlinear partial differential equations with discontinuous nonlinearities have been increasedly studied in the latest years because it arises in physics problems, as

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nonlinear elasticity theory or mechanics, and engineering topics. In this direction, concrete applications can be found in the books of Naniewicz–Panagiotopoulos [11].

The existence or multiplicity results for elliptic resonance problems with nonsmooth potential

$$\begin{cases} -\Delta u \in \lambda_k u + [g^-(x, u(x)), g^+(x, u(x))], & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N (N \geq 3)$ be a nonempty bounded open subset with a smooth boundary $\partial\Omega$, λ_k, $g^-(x,t)$ and $g^+(x,t)$ is defined in Sect. 3, have been obtained by many authors through using the nonsmooth critical point theory, see [4,5,7–10,12,15] and the references therein. However, in order to obtain the solutions of the problem (1), our technique is firstly need to prove the following elliptic problems with discontinuous nonlinearities

$$\begin{cases} -\Delta u \in \frac{1}{\lambda}[\lambda_k u + [g^-(x, u(x)), g^+(x, u(x))]] & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\lambda \in (0, 1]$, has infinitely many solutions. secondly, by taking the limit with respect to λ for the problem (2), we obtain problem (1) has two nontrivial solutions.

Our approach is based on nonsmooth critical point theory for locally Lipschitz functionals, as this was originally formulated by Chang [1]. For the convenience of the reader, below we recall some basic definitions and facts from this theory.

Let E be a real Banach space and E^* its conjugate space, we denote by $\langle \cdot, \cdot \rangle$ the dulity pairing between E^* and E. A function $\varphi : E \to R$ is called locally Lipschitz if for each $u \in E$ there exists a neighborhood U of u and a constant $L \geq 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L\|x - y\| \quad \forall x, y \in U.$$

For a locally Lipschitz function $\varphi : E \to R$, we define the generalized directional derivative of φ at point u in the direction $h \in E$ as

$$\varphi^\circ(u; h) = \overline{\lim_{v \to 0, s \downarrow 0}} \frac{1}{s}[\varphi(u + v + sh) - \varphi(u + v)].$$

The generalized gradient of the locally Lipschitz function φ at the point u, denoted by $\partial\varphi(u)$, is the set

$$\partial\varphi(u) = \{w \in E^* : \langle w, v \rangle \leq \varphi^0(u; v), \forall v \in E\}.$$

If $\varphi \in C^1(E)$, then $\partial\varphi(u) = \{\varphi'(u)\}$ for all $u \in E$. We set $m(u) = \min_{w \in \partial\varphi(u)} \|w\|$. Then the function $m(u)$ exists and is lower semi-continuous. A point $u \in E$ is said to be a critical point of the locally Lipschitz function $\varphi : E \to R$, if $0 \in \partial\varphi(u)$. If $u \in E$ is a critical point, the value $c = \varphi(u)$ is said to be a critical value of φ.

For more details, we refer to [1–3] for the properties of the generalized directional derivative and the generalized gradient.

The critical point theory for smooth functionals uses a compactness condition known as the Palais-Smale condition (PS). In the present nonsmooth setting this condition takes the following form:

A locally Lipschitz function $\varphi : E \to R$ is said to satisfy the non-smooth PS-condition, if any sequence $\{x_n\} \subseteq E$ with $\{\varphi(x_n)\}$ is bounded and $m(x_n) \to 0$ has a strongly convergent subsequence.

During the proof of our main results, we will use some basic concepts and properties from set-valued analysis, for convenience, we list them as follows:

Definition 1 ([6]) Suppose X, Y are Hausdorff topological spaces. Let $F : X \to 2^Y$ be a multifunction. We say that F is upper semicontinuous at x_0, if for any open subset $V \subset Y$ with $F(x_0) \subset V$, there exists $U \in N(x_0)$, such that $F(U) \subset V$. If F is upper semicontinuous at every $x_0 \in X$, we say that F is upper semicontinuous.

Proposition 1 ([6]) *If $\varphi : X \to R$ is a locally Lipschitz function, then the multifunction $x \to \partial\varphi(x)$ is upper semicontinuous from X into $X^*_{w^*}$.*

The paper is arranged as follows. In Sect. 2, we establish the double linking theorem for nonsmooth locally Lipschitz functions. In Sect. 3, we are devoted to multiplicity results for elliptic resonance problems with discontinuous nonlinearities.

2 Abstract results

Let E be a reflexive Banach space. Define a class of contractions of E as follows: $\Phi = \{\Gamma(\cdot,\cdot) \in C([0,1] \times E, E) : \Gamma(0,\cdot) = id$; for each $t \in [0,1]$, $\Gamma(t,\cdot)$ is a homeomorphism of E onto itself and $\Gamma^{-1}(\cdot,\cdot)$ is continuous on $[0,1) \times E$; there exists a $x_0 \in E$ such that $\Gamma(1,x) = x_0$ for each $x \in E$ and that $\Gamma(t,x) \to x_0$ as $t \to 1$ uniformly on bounded subset of $E\}$.

The following concepts of linking and double linking were introduced by Schechter-Tintarev[13] and Schechter-Zou [14], respectively.

Definition 2 [14] A subset A of E is linked to a subset B of E if $A \bigcap B = \emptyset$ and for every $\Gamma \in \Phi$, there is a $t \in [0,1]$ such that $\Gamma(t,A) \bigcap B \neq \emptyset$.

Definition 3 [14] Let $A, B \subset E$ be two closed subsets, if A and B link each other, we call them double linking.

A typical example of double linking is the following ([14]):

Let $E = M \bigoplus N$, where M, N are closed subspaces with one of them finite dimensional. If $y_0 \in M\backslash\{0\}$ and $0 < \rho < R$, then the sets

$$A = \{u = v + sy_0 : v \in N,\ s \geq 0,\ \|u\| = R\} \bigcup \left[N \bigcap \bar{B}_R\right]$$

and

$$B = M \bigcap \partial B_\rho$$

link each other in the sense of definition 3, where $B_r = \{u \in E : \|u\| < r\}$.

The purpose of this section is to establish the existence of two bounded nonsmooth Palais-Smale sequences from the double linking which yield either two critical points with different critical values or two critical points in two nonintersection sets.

Let $G_\lambda(u) = \lambda I(u) - J(u), \lambda \in \Lambda \subset (0,+\infty)$. $I \in C^1(X,R)$ and $J : E \to R$ is a locally Lipschitz functional and they map bounded set into bounded set, respectively.

Assume that

(H_1) $I(u) \geq 0$ for all $u \in E$ and either $I(u) \to \infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.
(H_2) $I(u) \leq 0$ for all $u \in E$ and either $I(u) \to -\infty$ or $|J(u)| \to \infty$ as $\|u\| \to \infty$.

Furthermore, we suppose that

(H_3) $a_0(\lambda) = \sup_A G_\lambda \leq b_0(\lambda) = \inf_B G_\lambda$ for any $\lambda \in \Lambda$.

In this section, our main results are the following theorem:

Theorem 1 *Assume that* (H_1) *(or* (H_2)*) and* (H_3) *hold,*

(1) *If* A *links* B *and* A *is bounded, then for almost all* $\lambda \in \Lambda$ *there exists* $u_k(\lambda) \in E$ *such that* $\sup_k \|u_k(\lambda)\| < \infty$, $m_\lambda(u_k(\lambda)) \to 0$ *and*

$$G_\lambda(u_k(\lambda)) \to a(\lambda) = \inf_{\Gamma \in \Phi} \sup_{s \in [0,1], u \in A} G_\lambda(\Gamma(s,u)), \ k \to \infty.$$

Furthermore, if $a(\lambda) = b_0(\lambda)$ *then* $dist(u_k(\lambda), B) \to 0, k \to \infty$.

(2) *If* B *links* A *and* B *is bounded, then for almost all* $\lambda \in \Lambda$ *there exists* $v_k(\lambda) \in E$ *such that* $\sup_k \|v_k(\lambda)\| < \infty$, $m_\lambda(v_k(\lambda)) \to 0$ *and*

$$G_\lambda(v_k(\lambda)) \to b(\lambda) = \inf_{\Gamma \in \Phi} \sup_{s \in [0,1], v \in B} G_\lambda(\Gamma(s,v)), \ k \to \infty.$$

Furthermore, if $b(\lambda) = a_0(\lambda)$ *then* $dist(v_k(\lambda), A) \to 0, k \to \infty$.

For the reader's convenience, we give the rough proof.

Proof (1) First, we prove that conclusion (1) with the first alternative (H_1) is true. Evidently, $a(\lambda) \geq b_0(\lambda)$ since A links B. By (H_3), the map $\lambda \to a(\lambda)$ is nondecreasing, hence, $a'(\lambda)$ exists for almost all $\lambda \in \Lambda$. For fixed $\lambda \in \Lambda$, let $\lambda_n \in (\lambda, 2\lambda) \bigcap \Lambda$ satisfy $\lambda_n \to \lambda$ as $n \to \infty$, then there exists $\bar{n}(\lambda)$ such that

$$a'(\lambda) - 1 \leq \frac{a(\lambda_n) - a(\lambda)}{\lambda_n - \lambda} \leq a'(\lambda) + 1$$

for $n \geq \bar{n}(\lambda)$.

As in the proof of Theorem 2.1 in [14], we can obtain that:

(1^0) there exist $\Gamma_n \in \Phi$, $k_0 = k_0(\lambda) > 0$ such that

$$\|\Gamma_n(s,u)\| \leq k_0$$

whenever

$$G_\lambda(\Gamma_n(s,u)) \geq a(\lambda) - (\lambda_n - \lambda)$$

(2^0)

$$G_\lambda(\Gamma_n(s,u)) \leq G_{\lambda_n}(\Gamma_n(s,u)) \leq a(\lambda) + (a'(\lambda) + 2)(\lambda_n - \lambda); \tag{3}$$

(3^0) We consider the case of $a(\lambda) > b_0(\lambda)$. For $\varepsilon \in \left(0, \frac{a(\lambda) - b_0(\lambda)}{2}\right)$, we define

$$Q_\varepsilon(\lambda) = \{u \in E : \|u\| \leq k_0 + 4, \ \ |G_\lambda(u) - a(\lambda)| \leq 4\varepsilon\}. \tag{4}$$

Then we can obtain that $Q_\varepsilon(\lambda) \neq \emptyset$ (see Step 3 of Theorem 2.1 in [14]).

Next, we claim that there exists $u \in Q_\varepsilon(\lambda)$ such that $m_\lambda(u) < \varepsilon$ for $\varepsilon \in \left(0, \frac{a(\lambda) - b_0(\lambda)}{2}\right)$ small enough. By negation, we assume that there exists $\varepsilon_0 \in \left(0, \frac{a(\lambda) - b_0(\lambda)}{2}\right)$ such that $m_\lambda(u_0) \geq 3\varepsilon_0$ for all $u_0 \in Q_{\varepsilon_0}(\lambda)$, hence there exists a $w_0 \in \partial G_\lambda(u_0)$ such that $\|w_0\| = m_\lambda(u_0)$, then $B_{\|w_0\|} \bigcap \partial G_\lambda(u_0) = \emptyset$, where B_r is the ball centered at θ with radius r in E^*. Due to the separation theorem for convex sets, there exists $h_0 \in E$ such that $\|h_0\| = 1$,

$$\langle x^*, h_0 \rangle \geq \langle w_0, h_0 \rangle \geq \langle w, h_0 \rangle$$

for each $w \in B_{\|w_0\|}$ and each $x^* \in \partial G_\lambda(u_0)$. By the Hahn-Banach theorem,

$$\max_{w \in \overline{B}_{\|w_0\|}} \langle w, h_0 \rangle = \|w_0\| \cdot \|h_0\| = \|w_0\|$$

then we have

$$\langle x^*, h_0 \rangle \geq \|w_0\| = m_\lambda(u_0) \geq 3\varepsilon_0 > 2\varepsilon_0, \quad \forall x^* \in \partial G_\lambda(u_0).$$

Since the mapping $x \to \partial G_\lambda(x)$ is weak* upper semi-continuous, there exists an open neighborhood $N(u_0) \subset B(u_0, \delta) = \{u \in E : \|u - u_0\| \leq \delta\}$ of u_0 such that for each $u \in N(u_0)$, we have

$$\langle x^*, h_0 \rangle > 2\varepsilon_0$$

for each $x^* \in \partial G_\lambda(u)$. The set of all such neighborhoods covers $Q_{\varepsilon_0}(\lambda)$. Therefore there exists a local refinement $\{N_i : i \in \Gamma\}$, where Γ denotes the index set. Thus, for every $i \in \Gamma$, there exists a $u_i \in Q_{\varepsilon_0}(\lambda)$ such that $N_i \subset N(u_i)$. Let $\rho_i(u)$ denote the distance from u to the complement of N_i, then ρ_i is Lipschitz continuous and vanishes outside N_i. Let

$$\beta_i(u) = \frac{\rho_i}{\Sigma_j \rho_j(u)}$$

and let

$$v_\lambda(u) = \Sigma \beta_i(u) h_i.$$

Then $v_\lambda : Q_{\varepsilon_0}(\lambda) \to E$ is local Lipschitz and satisfies:

$$\|v_\lambda(u)\| \leq 1,$$

$$\langle x^*, v_\lambda(u) \rangle \geq 2\varepsilon_0$$

for

$$x^* \in \partial G_\lambda(u), \quad u \in Q_{\varepsilon_0}(\lambda).$$

We take n so large that

$$(a'(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon_0, \quad \lambda_n - \lambda < \varepsilon_0.$$

Define

$$Q^*_{\varepsilon_0}(\lambda) = \{u \in E : \|u\| \leq k_0 + 1, \quad a(\lambda) - (\lambda_n - \lambda) \leq G_\lambda(u) \leq a(\lambda) + \varepsilon_0\}, \tag{5}$$

by (4), similar reasoning show that $Q^*_{\varepsilon_0}(\lambda) \neq \emptyset$ and $Q^*_{\varepsilon_0}(\lambda) \subset Q_{\varepsilon_0}(\lambda)$.

Define

$$\Omega_1 = \{u \in E : \|u\| \leq k_0 + 1\}$$
$$\Omega_2 = \{u \in E : \|u\| \geq k_0 + 2\}$$

$$\Omega_3 = \{u \in E : \text{either } G_\lambda(u) < a(\lambda) - (\lambda_n - \lambda) \text{ or } G_\lambda(u) > a(\lambda) + 2\varepsilon_0\}$$
$$\Omega_4 = \{u \in E : a(\lambda) - (\lambda_n - \lambda) \leq G_\lambda(u) \leq a(\lambda) + \varepsilon_0\}$$

Define

$$\xi(u) = \frac{\mathtt{dist}(u, \Omega_2)}{\mathtt{dist}(u, \Omega_2) + \mathtt{dist}(u, \Omega_1)},$$
$$\zeta(u) = \frac{\mathtt{dist}(u, \Omega_3)}{\mathtt{dist}(u, \Omega_3) + \mathtt{dist}(u, \Omega_4)},$$

where $\mathtt{dist}$ denotes the distance functional.

Therefore,

$$V_\lambda(u) = \xi(u)\zeta(u)v_\lambda(u)$$

is a locally Lipschitz continuous vector field from E to E. Moreover,

$$\begin{aligned} &\|V_\lambda\| \le 1, && \forall u \in E,\\ &\langle x^*, V_\lambda(u)\rangle > 2\varepsilon_0, && \forall u \in Q^*_{\varepsilon_0}(\lambda)\ \ x^* \in \partial G_\lambda(u),\\ &\langle x^*, V_\lambda(u)\rangle \ge 0, && \forall u \in E. \end{aligned}$$

Consider the initial boundary value problem:

$$\frac{d\eta(t,u)}{dt} = -V_\lambda(\eta(t,u)), \quad \eta(0,u) = u.$$

By a well known existence theorem for ordinary differential equation in a Banach space there exists a unique continuous solution $\eta(t, u)$ such that $G_\lambda(\eta(t, u))$ is non-increasing in t. In fact,

$$\begin{aligned} \frac{d}{dt}G_\lambda(\eta(t,u)) &\le \max\left\{\left\langle w, \frac{d}{dt}\eta(t,u)\right\rangle : w \in \partial G_\lambda(\eta(t,u))\right\}\\ &= -\min\{\langle w, -V_\lambda(\eta(t,u))\rangle : w \in \partial G_\lambda(\eta(t,u))\}\\ &\le \begin{cases} -2\varepsilon_0, & \eta(t,u) \in Q^*_{\varepsilon_0}(\lambda);\\ 0, & \text{otherwise.}\end{cases} \end{aligned}$$

Define

$$\tilde\Gamma(s,u) = \begin{cases} \eta(2s,u), & 0 \le s \le \frac{1}{2};\\ \eta(1, \Gamma_n(2s-1,u)), & \frac{1}{2} \le s \le 1. \end{cases}$$

Then it is easy to check that $\tilde\Gamma \in \Phi$, we want to prove that

$$G_\lambda(\tilde\Gamma(s,u)) \le a(\lambda) - (\lambda_n - \lambda), \ \ \forall s \in [0,1], \ u \in A.$$

which provides the desired contradiction. Choose any $u \in A$, if $0 \le s \le \frac{1}{2}$, by (H_3) and the choice of ε_0, we get that

$$\begin{aligned} G_\lambda(\tilde\Gamma(s,u)) &= G_\lambda(\eta(2s,u)) \le G_\lambda(u) \le a_0(\lambda) < b_0(\lambda) \le a(\lambda) - 2\varepsilon_0\\ &\le a(\lambda) - (\lambda_n - \lambda). \end{aligned} \tag{6}$$

If $\frac{1}{2} \le s \le 1$, then $\tilde\Gamma(s,u) = \eta(1, \Gamma_n(2s-1,u))$, if $G_\lambda(\Gamma_n(2s-1,u)) < a(\lambda) - (\lambda_n - \lambda)$, for $\frac{1}{2} \le s \le 1$, then

$$\begin{aligned} G_\lambda(\tilde\Gamma(s,u))) &= G_\lambda(\eta(1, \Gamma_n(2s-1,u)))\\ &\le G_\lambda(\eta(0, \Gamma_n(2s-1,u)))\\ &= G_\lambda(\Gamma_n(2s-1,u))\\ &< a(\lambda) - (\lambda_n - \lambda). \end{aligned} \tag{7}$$

If there exists $s_0 \in \left[\frac{1}{2}, 1\right]$ such that $G_\lambda(\Gamma_n(2s_0-1,u)) \geq a(\lambda)-(\lambda_n-\lambda)$, then by (1°), $\|\Gamma_n(2s_0-1,u)\| \leq k_0$. By (3) and (5), we see that $\Gamma_n(2s_0-1,u) \in Q^*_{\varepsilon_0}(\lambda)$. Since

$$\|\eta(t,\Gamma_n(2s_0-1,u))-\Gamma_n(2s_0-1,u)\| \leq \left\| \int_0^t \frac{d\eta(\sigma,\Gamma_n(2s_0-1,u))}{d\sigma} d\sigma \right\| \leq t,$$

it follows that

$$\|\eta(t,\Gamma_n(2s_0-1,u))\| \leq \|\Gamma_n(2s_0-1,u)\| + t \leq k_0+1, \quad \forall t \in [0,1].$$

Two cases again:

If $G_\lambda(\eta(t_0,\Gamma_n(2s_0-1,u))) < a(\lambda)-(\lambda_n-\lambda)$ for some $t_0 \in [0,1]$, then

$$G_\lambda(\tilde{\Gamma}(s_0,u)) = G_\lambda(\eta(1,\Gamma_n(2s_0-1,u))) < a(\lambda)-(\lambda_n-\lambda). \tag{8}$$

On the other hand, by (3), we see that

$$\begin{aligned} a(\lambda)-(\lambda_n-\lambda) &\leq G_\lambda(\eta(1,\Gamma_n(2s_0-1,u))) \\ &\leq G_\lambda(\eta(t,\Gamma_n(2s_0-1,u))) \\ &\leq G_\lambda(\Gamma_n(2s_0-1,u)) \\ &\leq a(\lambda)+\varepsilon_0, \end{aligned}$$

for all $t \in [0,1]$. Thus, $\eta(t,\Gamma_n(2s_0-1,u)) \in Q^*_{\varepsilon_0}(\lambda)$, for all $t \in [0,1]$. Since $\langle x^*, V_\lambda(u)\rangle > 2\varepsilon_0$, $\forall u \in Q^*_{\varepsilon_0}(\lambda)$, $x^* \in \partial G_\lambda(u)$, we have

$$\begin{aligned} &G_\lambda(\eta(t,\Gamma_n(2s_0-1,u))) - G_\lambda(\Gamma_n(2s_0-1,u)) \\ &= \int_0^t \frac{dG_\lambda(\eta(\sigma,\Gamma_n(2s_0-1,u)))}{d\sigma} d\sigma \\ &\leq \int_0^t \max\{\langle w, V_\lambda(\eta(\sigma,\Gamma_n(2s_0-1,u)))\rangle : \\ &\quad w \in \partial G_\lambda(\eta(\sigma,\Gamma_n(2s_0-1,u)))\} d\sigma \\ &\leq -\int_0^t \min\{\langle w, -V_\lambda(\eta(\sigma,\Gamma_n(2s_0-1,u)))\rangle : \\ &\quad w \in \partial G_\lambda(\eta(\sigma,\Gamma_n(2s_0-1,u)))\} d\sigma \\ &\leq -2\varepsilon_0 t. \end{aligned}$$

Therefore, when $t=1$, we have that

$$\begin{aligned} G_\lambda(\tilde{\Gamma}(s_0,u)) &= G_\lambda(\eta(1,\Gamma_n(2s_0-1,u))) \\ &\leq G_\lambda(\Gamma_n(2s_0-1,u)) - 2\varepsilon_0 \\ &\leq a(\lambda)-(\lambda_n-\lambda). \end{aligned} \tag{9}$$

Combining (6), (7), (8) and (9), we get

$$G_\lambda(\tilde{\Gamma}(s,u)) \leq a(\lambda)-(\lambda_n-\lambda), \quad \forall s \in [0,1], \quad u \in A,$$

which contradicts the definition of $a(\lambda)$. This implies that the above claim in case of $a(\lambda) > b_0(\lambda)$. Evidently, the claim yields the conclusion (1) of this theorem.

(4^0) We prove that the conclusion (1) of theorem 2.1 is still true in case of $a(\lambda) = b_0(\lambda)$.

Since A is bounded, $d_A = \max\{\|u\| : u \in A\} < \infty$. For $\varepsilon > 0$, $T > 0$, we define

$$\bar{Q}(\varepsilon, T, \lambda) = \{u \in E : \|u\| \leq k_0 + 4 + d_A,\ \ |G_\lambda(u) - a(\lambda)| \leq 3\varepsilon,\ \ \mathtt{dist}(u, B) \leq 4T\}.$$

Then, $\bar{Q}(\varepsilon, T, \lambda) \neq \emptyset$ (see the Step 4 of Theorem 2.1 in [14]).

We prove that for ε and T small enough, there exists $u \in \bar{Q}(\varepsilon, T, \lambda)$ such that $m_\lambda(u) < \varepsilon$. If not, there exists $\delta > 0$, $\varepsilon_1 > 0$ and $T_1 \in (0, 1)$ such that

$$m_\lambda(u) \geq 3\delta, \quad \forall u \in \bar{Q}(\varepsilon_1, T_1, \lambda).$$

Define

$$\begin{aligned}\bar{Q}^*(\varepsilon_1, T_1, \lambda) = \{u \in E : \|u\| \leq k_0 + 3 + d_A,\ a(\lambda) - (\lambda_n - \lambda) \leq G_\lambda(u) \leq a(\lambda) + 3\varepsilon_1,\\ \mathtt{dist}(u, B) \leq 3T_1\}.\end{aligned}$$

The same reasoning shows that $\bar{Q}^*(\varepsilon_1, T_1, \lambda) \neq \emptyset$ and $\bar{Q}^*(\varepsilon_1, T_1, \lambda) \subset \bar{Q}(\varepsilon_1, T_1, \lambda)$. Let n so large that $(\lambda_n - \lambda) < \varepsilon_1$, $(a'(\lambda) + 2)(\lambda_n - \lambda) < \varepsilon_1$ and $(\lambda_n - \lambda) < \delta T_1$. As the proof of ($3^0$), we can construct a locally Lipschitz continuous mapping $\bar{V_\lambda}(u)$ on E such that

$$\begin{array}{ll}\|\bar{V_\lambda}\| \leq 1 & \forall u \in E,\\ \langle x^*, \bar{V_\lambda}(u)\rangle > 2\delta, & \forall u \in \bar{Q}^*(\varepsilon_1, T_1, \lambda) \quad x^* \in \partial G_\lambda(u),\\ \langle x^*, \bar{V_\lambda}(u)\rangle \geq 0 & \forall u \in E.\end{array}$$

Define

$$Q_1 = \{u \in E : \|u\| \leq k_0 + 2 + d_A,\ \ |G_\lambda(u) - a(\lambda)| \leq 2\varepsilon_1,\ \ \mathtt{dist}(u, B) \leq 2T_1\}.$$

As the proof of (4^0), $Q_1 \neq \emptyset$ and $Q_1 \subset \bar{Q}(\varepsilon_1, T_1, \lambda)$. Choose a Lipschitz continuous mapping γ from E into $[0, 1]$ which equals 1 on Q_1 and vanishes outside $\bar{Q}(\varepsilon_1, T_1, \lambda)$. Consider the following initial boundary value problem:

$$\frac{d\eta_1(t, u)}{dt} = -\gamma(\eta_1)V_\lambda(\eta_1), \quad \eta_1(0, u) = u.$$

By a well known existence theorem for ordinary differential equation in a Banach space there exists a unique continuous solution $\eta_1(t, u)$ such that $G_\lambda(\eta_1(t, u))$ is non-increasing in t. In fact,

$$\begin{aligned}\frac{d}{dt}G_\lambda(\eta_1(t, u)) &\leq \max\left\{\langle w, \frac{d}{dt}\eta_1(t, u)\rangle : w \in \partial G_\lambda(\eta_1(t, u))\right\}\\ &= -\min\{\langle w, -\gamma(\eta_1(t, u))V_\lambda(\eta_1(t, u))\rangle : w \in \partial G_\lambda(\eta_1(t, u))\}\\ &\leq \begin{cases}-2\delta\gamma(\eta_1(t, u)), & \eta_1(t, u) \in \bar{Q}^*(\varepsilon_1, T_1, \lambda);\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, $\frac{d}{dt}G_\lambda(\eta_1(t, u)) \leq 0$.

As the proof Theorem 2.1 in [14], we can obtain that $\eta_1(s, u) \notin B$ for all $s \in [0, T_1]$ and $u \in A$ and $\eta_1(T_1, \Gamma_n(2s - 1, u)) \notin B, \forall u \in A, s \in [\frac{1}{2}, 1]$.

In order to get the final contradiction, we define

$$\Gamma_1^*(s, u) = \begin{cases}\eta_1(2sT_1, u), & 0 \leq s \leq \frac{1}{2};\\ \eta_1(T_1, \Gamma_n(2s - 1, u)), & \frac{1}{2} \leq s \leq 1.\end{cases}$$

Then it is easy to check that $\Gamma_1^* \in \Phi$. However, $\Gamma_1^*(s, A) \bigcap B = \emptyset$ for all $s \in [0, 1]$. We get the final contradiction.

As the proof Theorem 2.1 in [14], we can complete the conclusion (1) with the second alternative (H_2).

As an immediate consequence, we have

Theorem 2 *Assume that (H_1)(or (H_2)) and (H_3) hold and that A, B are bounded sets which link each other. If for all $\lambda \in \Lambda$, any bounded non-smooth (PS)-sequence of G_α (i.e., $m_\lambda(u_k) \to 0$ and $\{G_\lambda(u_k)\}, \{u_k\}$ are bounded) possesses a convergent subsequence, then for almost all $\lambda \in \Lambda$, G_λ has two different critical points u_λ and v_λ satisfying*

$$G_\lambda(u_\lambda) = a(\lambda), \quad m_\lambda(u_\lambda) = 0; \quad G_\lambda(v_\lambda) = b(\lambda), \quad m_\lambda(v_\lambda) = 0.$$

Particularly, if $a(\lambda) = b(\lambda)$, then $u_\lambda \in B$, $v_\lambda \in A$.

3 Some applications

Let $\Omega \subset R^N (N \geq 3)$ be a nonempty bounded open subset with a smooth boundary $\partial\Omega$. In this section, we are concerned with the multiplicity of the solutions of the following nonlinear elliptic equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u \in \lambda_k u + [g^-(x, u(x)), g^+(x, u(x))], & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where the discontinuous nonlinearities term $g(x, t) : \Omega \times R \to R$ is a locally bounded measurable function, and $g^-(x, t)$ and $g^+(x, t)$ are defined by

$$g^-(x, t) = \lim_{\delta \to 0} \inf_{|\xi - t| < \delta} g(x, \xi), \quad g^+(x, t) = \lim_{\delta \to 0} \sup_{|\xi - t| < \delta} g(x, \xi).$$

Obviously, $g^-(x, t)$ and $g^+(x, t)$ are respectively lower semi-continuous and upper semi-continuous. Denoting by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots$ all distinct eigenvalues of the eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Let $\{E_k\}$ be the eigenspace corresponding to the eigenvalues $\lambda_k (k = 1, 2, \ldots)$ and $N_k = E_1 \bigoplus E_2 \bigoplus \cdots \bigoplus E_k$.

In this section, the letter c will be indiscriminately used to denote various constant when the exact values are irrelevant.

Theorem 3 *Suppose that $g(x, t)$ satisfies the following conditions:*

(A_1) *The function $g^-(x, t)$ and $g^+(x, t)$ are superposition measurable, i.e. if $u : \Omega \to R$ is measurable implies that $x \mapsto g(x, u(x))$ is measurable;*

(A_2) *there exist a constant c and $1 < p < \frac{N+2}{N-2} (N \geq 3)$ such that*

$$|g(x, t)| \leq c(1 + |t|^p)$$

for all $x \in \Omega$ and $t \in R$;

(A_3)

$$2G(x, t) \geq (\lambda_{k-1} - \lambda_k)|t|^2$$

for almost all $x \in \Omega$;

(A_4)

$$\lim_{u\in N_k,\|u\|\to\infty}\int_\Omega G(x,u)dx \to \infty;$$

(A_5) *there exists a eigenvalue* $\lambda_l < \lambda_k$ *such that*

$$\limsup_{|t|\to 0}\frac{2G(x,t)}{|t|^2} < \lambda_l - \lambda_k$$

uniformly for almost all $x \in \Omega$*;*
Then for almost all $\lambda \in \left(\frac{\lambda_l}{\lambda_k}, 1\right]$, *the problem*

$$\begin{cases} -\Delta u \in \frac{1}{\lambda}[\lambda_k u + [g^-(x,u(x)), g^+(x,u(x))]] & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{11}$$

has two nontrivial solutions. Particularly, the problem (11) has infinitely many solutions. If, in addition,

(A_6) *there exists* $\theta \in (1,2)$ *and a constant* $c > 0$ *such that*

$$\liminf_{|t|\to\infty}\frac{tg(x,t)-2G(x,t)}{|t|^\theta} > 0$$

uniformly for almost all $x \in \Omega$, *and*

$$tg(x,t) \le ct^2.$$

Then the problem (10) *has two nontrivial solutions.*

As usual, we find solutions of problem (11) as critical points of the functional G_λ defined by

$$G_\lambda(u) = \frac{\lambda}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2dx - \int_\Omega G(x,u)dx, \quad u \in H_0^1(\Omega),\ \lambda > 0.$$

Under the assumption of (A_1) and (A_2), we know that the functional $G_\lambda(u)$ is locally Lipschitz on $H_0^1(\Omega)$ (see [1]).

Lemma 1 $G_\lambda(u) \to -\infty$ *uniformly for* $\lambda \in (0,1]$ *as* $\|u\| \to \infty, u \in N_k$.

Proof It follows the fact

$$\begin{aligned} G_\lambda(u) &= \frac{\lambda}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2dx - \int_\Omega G(x,u)dx \\ &\le \frac{1}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2dx - \int_\Omega G(x,u)dx \\ &\le -\int_\Omega G(x,u)dx \end{aligned}$$

and the assumption (A_4), we can easily obtain the conclusion.

Lemma 2 $G_\lambda(u) \le 0$ *for all* $u \in N_{k-1}, \lambda \in (0,1]$.

Proof By the assumption (A_3), $\forall u \in N_{k-1}$, we have

$$\begin{aligned}
G_\lambda(u) &= \frac{\lambda}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx \\
&\le \frac{1}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx \\
&\le \frac{1}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2 dx - \frac{1}{2}\int_\Omega [\lambda_{k-1}-\lambda_k]|u|^2 dx \\
&\le \frac{1}{2}\|u\|^2 - \frac{\lambda_{k-1}}{2}\|u\|_2^2 \\
&\le 0
\end{aligned}$$

Lemma 3 *Under the assumptions of Theorem 3, there exist* $\rho_0 > 0$, $c_0 > 0$ *such that*

$$G_\lambda(u) \ge c_0 \ \ for \ \ \|u\| = \rho_0, \ u \in N_{k-1}^{\perp},$$

where $\lambda \in \left(\frac{\lambda_l}{\lambda_k}, 1\right]$.

Proof By the assumption (A_5), for small $\varepsilon > 0$, there exists $\delta > 0$ such that

$$G(x,t) \le \frac{1}{2}(\lambda_l - \lambda_k - \varepsilon)|t|^2, \ \ \forall |t| \le \delta.$$

By the assumption (A_2), we have

$$G(x,u) \le c|u|^{p+1}, \ \ \forall |u| \ge \delta.$$

Hence, $\forall u \in N_{k-1}^{\perp}$, we have

$$\begin{aligned}
G_\lambda(u) &= \frac{\lambda}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx \\
&= \frac{\lambda}{2}\sum_{j\ge k}\lambda_j\|u_j\|_2^2 - \frac{1}{2}\lambda_k\sum_{j\ge k}\|u_j\|_2^2 - \int_{\{x\in\Omega:|u(x)|\le\delta\}} G(x,u(x))dx \\
&\quad - \int_{\{x\in\Omega:|u(x)|\ge\delta\}} G(x,u(x))dx \\
&\ge \frac{1}{2}\sum_{j\ge k}(\lambda\lambda_j - \lambda_k)\|u_j\|_2^2 - \frac{1}{2}(\lambda_l-\lambda_k-\varepsilon)\sum_{j\ge k}\|u_j\|_2^2 + \frac{1}{2}(\lambda_l-\lambda_k-\varepsilon) \\
&\quad \times \int_{\{x\in\Omega:|u(x)|>\delta\}} |u(x)|^2 dx - \int_{\{x\in\Omega:|u(x)|>\delta\}} G(x,u(x))dx \\
&\ge \frac{1}{2}\sum_{j\ge k}(\lambda\lambda_j - \lambda_l + \varepsilon)\|u_j\|_2^2 - \frac{1}{2}(\lambda_k-\lambda_l+\varepsilon)\delta^{1-p} \\
&\quad \times \int_{\{x\in\Omega:|u(x)|>\delta\}} |u(x)|^{p+1} dx - \int_{\{x\in\Omega:|u(x)|>\delta\}} G(x,u(x))dx
\end{aligned}$$

$$\geq \min_{j\geq k}\left[\frac{\lambda\lambda_j-\lambda_l}{\lambda_j}\right]\sum_{j\geq k}\lambda_j\|u_j\|_2^2-c\|u\|^{p+1}$$

$$=\min_{j\geq k}\left[\frac{\lambda\lambda_j-\lambda_l}{\lambda_j}\right]\|u\|^2-c\|u\|^{p+1}.$$

Hence, we can find $\rho_0>0$ and $c_0>0$ such that

$$G_\lambda(u)\geq c_0>0,\ \forall u\in N_{k-1}^{\perp}\ \text{ with }\ \|u\|=\rho_0\ \text{ for }\ \lambda\in\left(\frac{\lambda_l}{\lambda_k},1\right].$$

Lemma 4 *For each $\lambda\in\Lambda\subset(0,+\infty)$, any bounded nonsmooth P-S sequence of $G_\lambda(u)$ possesses a convergent subsequence.*

Proof Assume $\{u_n\}\subset H_0^1(\Omega)$ bounded such that $\{G_\lambda(u_n)\}$ is bounded and $m_\lambda(u_n)\to 0$, where $m_\lambda(u_n)=\min_{w\in\partial G_\lambda(u_n)}\|w\|$, We will show that $\{u_n\}$ has a convergent subsequence.

Set $\varphi(u)=\int_\Omega G(x,u(x))dx$. Since $\partial G_\lambda(u_n)\subset H_0^1(\Omega)$ is a weak* compact set and the norm in Banach space is weakly lower semi-continuous, Thus, there exists $w_n\in\partial G_\lambda(u_n)$ such that $m_\lambda(u_n)=\|w_n\|$.

Define $A: H_0^1(\Omega)\to H_0^1(\Omega)$ by

$$\langle Au,v\rangle=\int_\Omega(\nabla u,\nabla v)_{R^N}dx$$

for all $u\in H_0^1(\Omega)$, $v\in H_0^1(\Omega)$. Here $\langle\cdot,\cdot\rangle$ is the duality pairing between $(H_0^1(\Omega))^*$ and $H_0^1(\Omega)$, $(\cdot,\cdot)_{R^N}$ denote the inner product in R^N.

Since $\partial G_\lambda(u)\subset\lambda Au-\lambda_k u-\partial\varphi(u)$, $w_n=\lambda Au_n-\lambda_k u_n+v_n$, where $v_n\in\partial\varphi(u_n)$ and

$$\langle w_n,v\rangle=\lambda\int_\Omega(\nabla u_n,\nabla v)_{R^N}dx-\lambda_k\int_\Omega(u_n,v)_{R^N}dx+\langle v_n,v\rangle.$$

Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, passing if necessary to a subsequence, we may assume that $u_n\rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n\to u$ strongly in $L^q(\Omega)$, $1<q<\frac{2N}{N-2}$, and $u_n(x)\to u(x)$ a.e. in Ω. By the definition of A, it is easy to verify that A is monotone; moreover, A is demi-continuous, in fact, let $u_n\to u$ in $H_0^1(\Omega)$, then for every $v\in H_0^1(\Omega)$ we have

$$|\langle Au_n-Au,v\rangle|=\left|\int_\Omega(\nabla u_n-\nabla u,\nabla v)_{R^N}dx\right|$$

$$=\left|\int_\Omega(\nabla u_n,\nabla v)_{R^N}dx-\int_\Omega(\nabla u,\nabla v)_{R^N}dx\right|.$$

Since $u_n\to u$ in $H_0^1(\Omega)$, $\nabla u_n\to\nabla u$ in $L^2(\Omega)$. Hence,

$$\int_\Omega(\nabla u_n,\nabla v)_{R^N}dx\to\int_\Omega(\nabla u,\nabla v)_{R^N}dx,$$

that is

$$|\langle Au_n-Au,v\rangle|\to 0.$$

Because $v \in H_0^1(\Omega)$ is arbitrary, it follows that $Au_n \rightharpoonup Au$ in $H_0^1(\Omega)$, and so, A is demi-continuous. Hence, A is maximal monotone.

Since

$$\left[\lambda\langle Au_n, u_n - u\rangle - \lambda_k \int_\Omega (u_n, u_n - u)_{R^N} dx + \langle v_n, u_n - u\rangle\right] = \langle w_n, u_n - u\rangle.$$

Thus, we have

$$\limsup\langle Au_n, u_n - u\rangle = \limsup \frac{1}{\lambda}\left[\lambda_k \int_\Omega (u_n, u_n - u)_{R^N} dx + \langle w_n, u_n - u\rangle \right.$$
$$\left. + \langle v_n, u - u_n\rangle \right].$$

According to the definition of the generalized directional derivative, by the assumption (A_1) and Hölder inequality, we have

$$\begin{aligned}
\langle v_n, u - u_n\rangle &= \varphi^0(u_n; u - u_n) \\
&= \int_{\{x\in\Omega|(u-u_n)>0\}} (u - u_n) g^+(x, u_n(x)) dx \\
&\quad + \int_{\{x\in\Omega|(u-u_n)<0\}} (u - u_n) g^-(x, u_n(x)) dx \\
&\le \int_\Omega |u_n - u| g^+(x, u_n(x)) dx \\
&\le c\int_\Omega |u_n - u||u_n|^p dx + c\int_\Omega |u_n - u| dx \\
&\le c\|u_n - u\|_{\frac{2N}{2N-p(N-2)}} \|u_n\|^p_{\frac{2N}{N-2}} + c\|u_n - u\|_2
\end{aligned}$$

Recall that we have used the fact that

$$\begin{aligned}
g^+(x, u_n(x)) &= \lim_{\delta\to 0} \sup_{|\xi - u_n|<\delta} g(x, \xi) \\
&\le \lim_{\delta\to 0} \sup_{|\xi - u_n|<\delta} [c|\xi|^p + c] \\
&\le c \lim_{\delta\to 0} \sup_{|\xi - u_n|<\delta} |\xi|^p + c \\
&= c \lim_{\delta\to 0} \sup_{|\eta|<\delta} |u_n + \eta|^p + c \\
&= c|u_n|^p + c.
\end{aligned}$$

Hence, by the embedding theorem and $1 < p < \frac{N+2}{N-2}$, we have

$$\limsup\langle Au_n, u_n - u\rangle \le 0.$$

Because $\{Au_n\} \subset H_0^1(\Omega)$ is bounded and so we may assume that $Au_n \rightharpoonup v$ in $H_0^1(\Omega)$. Since A is maximal monotone, it has the property(M) (see [16, p. 583]). Therefore, we have

$v = Au$ and $\langle Au_n, u_n\rangle \to \langle Au, u\rangle$, that is $\|\nabla u_n\|_2 \to \|\nabla u\|_2$, also because $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$, Hence, $u_n \to u$ in $H_0^1(\Omega)$.

Proof of Theorem 3 By the Lemma 1, 2 and 3, there exists $R_0 > \rho_0 > 0$ such that

$$a_0(\lambda) = \sup_A G_\lambda(u) \le 0 \le c_0 \le b_0(\lambda) = \inf_B G_\lambda(u)$$

for $\lambda \in (\frac{\lambda_l}{\lambda_k}, 1]$, where

$$A = \{u = v + sy : v \in N_{k-1}, s \ge 0, \|u\| = R_0\} \bigcup \left[N_{k-1} \bigcap \bar{B}_{R_0}\right],$$

with $y_0 \in E_k$ with $\|y_0\| = 1$, and

$$B = \{u \in N_{k-1}^{\perp} : \|u\| = \rho_0\}.$$

Theorem 2 implies, for almost all $\lambda \in (\frac{\lambda_l}{\lambda_k}, 1]$, that there are two different critical points u_λ, v_λ satisfying

$$G_\lambda(u_\lambda) = a(\lambda) \ge b_0(\lambda) \ge c_0, \quad m_\lambda(u_\lambda) = 0,$$

$$G_\lambda(v_\lambda) = b(\lambda) \le a_0(\lambda) \le 0, \quad m_\lambda(v_\lambda) = 0.$$

This is the first part of Theorem 3. For the second part of the Theorem 3, we choose $\lambda^n \to 1$, and u_n such that $G_{\lambda^n}(u_n) = a_{\lambda^n}, m_{\lambda^n}(u_n) = 0$. We claim that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Next, we prove the claim.

Since $\partial G_{\lambda^n}(u_n) \subset H_0^1(\Omega)$ is a weak* compact set and the norm in Banach space is weakly lower semi-continuous, Thus, there exists $w_n \in \partial G_{\lambda^n}(u_n)$ such that $m_{\lambda^n}(u_n) = \|w_n\|$ and

$$\langle w_n, y\rangle = \lambda^n \int_\Omega (\nabla u_n, \nabla y)dx - \lambda_k \int_\Omega u_n(x)y(x)dx - \langle v_n, y\rangle, \ \ \forall y \in H_0^1(\Omega)$$

where $v_n \in \partial\varphi(u_n)$ and

$$\langle v_n, y\rangle = \int_\Omega p_n(x)y(x)dx, \quad p_n(x) \in [g^-(x, u_n(x)), g^+(x, u_n(x))].$$

By the assumption (A_6), the proof is the same as the lemma 3.3 in [15], we can obtain that there exist $\delta > 0$ and constant $c > 0$ such that

$$tg^-(x,t) - 2G(x,t) > c|t|^\theta, \ \ \forall\, t \ge \delta$$

and

$$tg^+(x,t) - 2G(x,t) > c|t|^\theta, \ \ \forall\, t \le -\delta.$$

Hence,

$$\begin{aligned}
c+\|u_n\| &\geq \langle w_n, u_n\rangle - 2\int_\Omega G(x,u_n(x))dx \\
&= \int_\Omega [p_n(x)u_n(x) - 2G(x,u_n(x))]dx \\
&\geq \int_{\{u_n(x)>0\}} [g^-(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&\quad + \int_{\{u_n(x)<0\}} [g^+(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&= \int_{\{u_n(x)>\delta\}} [g^-(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&\quad + \int_{\{u_n(x)<-\delta\}} [g^+(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&\quad + \int_{\{0<u_n(x)\leq\delta\}} [g^-(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&\quad + \int_{\{0>u_n(x)\geq-\delta\}} [g^+(x,u_n(x))u_n(x) - 2G(x,u_n(x))]dx \\
&\geq c\int_{\{|u_n|>\delta\}} |u_n(x)|^\theta dx - c.
\end{aligned}$$

Taking $s=\frac{(2-\theta)(N-2)}{2N+4-N\theta}$, then $s\in(0,1)$, and by the Hölder inequality, we have

$$\begin{aligned}
\int_{\{|u_n|>\delta\}} |u_n(x)|^2 dx &= \int_{\{|u_n|>\delta\}} |u_n(x)|^{2(1-s)}|u_n(x)|^{2s} dx \\
&\leq \left(\int_{\{|u_n|>\delta\}} |u_n|^\theta dx\right)^{\frac{2(1-s)}{\theta}} \left(\int_{\{|u_n|>\delta\}} |u_n|^{\frac{2N+4}{N}} dx\right)^{\frac{2sN}{2N+4}} \\
&\leq (c+c\|u_n\|)^{\frac{2(1-s)}{\theta}}\|u_n\|^{2s}.
\end{aligned}$$

It follows from the condition (A_6) that

$$\begin{aligned}
\lambda^n\|u_n\|^2 &= \lambda_k\int_\Omega |u_n(x)|^2dx + \langle w_n, u_n\rangle + \int_\Omega p_n(x)u_n(x)dx \\
&\leq \lambda_k\int_\Omega |u_n(x)|^2dx + \langle w_n, u_n\rangle + \int_{\{u_n(x)>0\}} g^-(x,u_n(x))u_n(x)dx \\
&\quad + \int_{\{u_n(x)<0\}} g^+(x,u_n(x))u_n(x)dx
\end{aligned}$$

$$\leq (\lambda_k + c)\left[\int_\Omega |u_n(x)|^2 dx\right] + \|u_n\|$$

$$= (\lambda_k + c)\left[\int_{\{|u_n|<\delta\}} |u_n(x)|^2 dx + \int_{\{|u_n|\geq\delta\}} |u_n(x)|^2 dx\right] + \|u_n\|$$

$$\leq c + \|u_n\| + (c + c\|u_n\|)^{\frac{2(1-s)}{\theta}} \|u_n\|^{2s}$$

Since $\theta \in (1, 2)$ and $s \in (0, 1)$, then $\frac{2(1-s)}{\theta} + 2s < 2$. Thus, we see that $\{\|u_n\|\}$ is bounded.

Applying Lemma 4, we know that there exists a $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$. In order to complete the conclusion, we should show that u satisfies $m_1(u) = 0$. In fact, from the proof in Lemma 4, we see that there exists $w_n \in \partial G_{\lambda^n}(u_n)$ such that $m_{\lambda^n}(u_n) = \|w_n\|$ and

$$\langle w_n, v\rangle = \lambda^n \int_\Omega (\nabla u_n, \nabla v)_{R^N} dx - \lambda_k \int_\Omega (u_n, v)_{R^N} dx + \langle v_n, v\rangle, \quad \forall\, v \in H_0^1(\Omega),$$

where $v_n \in \partial\varphi(u_n)$. Because $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and $m_{\lambda^n}(u_n) = 0$, it is easy to see that $\{v_n\}$ is bounded in $[H_0^1(\Omega)]^*$. Since $\partial\varphi(u_n)$ is w^*-compact, then, there exists $v^* \in [H_0^1(\Omega)]^*$ such that $v_n \to v^*$ (w^* convergence). Combing $u_n \to u$ in $H_0^1(\Omega)$ and Proposition 1, we can obtain that $v^* \in \partial\varphi(u)$. Hence, we have proved that

$$\begin{aligned} 0 \;&=\; \lambda^n \int_\Omega (\nabla u_n, \nabla v)_{R^N} dx - \lambda_k \int_\Omega (u_n, v)_{R^N} dx + \langle v_n, v\rangle \\ &\to \int_\Omega (\nabla u, \nabla v)_{R^N} dx - \lambda_k \int_\Omega (u, v)_{R^N} dx + \langle v^*, v\rangle \\ &= \langle Au - \lambda_k u - v^*, v\rangle. \end{aligned}$$

Obviously, $Au - \lambda_k u - v^* \in \partial G_1(u)$ and then $0 \in \partial G_1(u)$, that is, $m_1(u) = 0$. Accordingly, u is a nontrivial critical point of functional G_1 satisfying

$$G_1(u) \geq c_0, \quad m_1(u) = 0.$$

Similarly, we can obtain another nontrivial critical point of functional G_1 satisfying

$$G_1(v) \geq 0, \quad m_1(v) = 0.$$

It follows by a similar proof as in Theorem 3 that we can obtain the following theorem.

Theorem 4 *Suppose that $g(x, t)$ satisfies (A_1), (A_2), and the following conditions:*

(A_3') *there exists $m \in N$ satisfying $\lambda_m < \lambda_k$, such that*

$$2G(x, t) \geq (\lambda_{m-1} - \lambda_k)|t|^2$$

uniformly for almost all $x \in \Omega$;

(A_4)

$$\lim_{u \in N_m, \|u\| \to \infty} \int_\Omega G(x, u) dx \to \infty;$$

(A_5) *there exists a eigenvalue* $\lambda_l < \lambda_m$ *such that*

$$\limsup_{|t|\to 0} \frac{2G(x,t)}{|t|^2} < \lambda_l - \lambda_m$$

uniformly for almost all $x \in \Omega$*;*
Then for almost all $\lambda \in \left(\frac{\lambda_l}{\lambda_m}, 1\right]$*, the problem (11) has two nontrivial solutions. Particularly, the problem (11) has infinitely many solutions. If, in addition,*

(A_6) *there exists* $\theta \in (1,2)$ *and a constant* $c > 0$ *such that*

$$\liminf_{|t|\to\infty} \frac{tg(x,t) - 2G(x,t)}{|t|^\theta} > 0$$

uniformly for almost all $x \in \Omega$*, and*

$$tg(x,t) \le ct^2.$$

Then the problem (10) has two nontrivial solutions.

Theorem 5 *Suppose that* $g(x,t)$ *satisfies* (A_1) *and the following conditions:*

(B_1) *there exists* $m \in N$ *satisfying* $\lambda_m > \lambda_k$*, and* $a(x) \in L^\infty(\Omega)$ *such that*

$$\lambda_m \le \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \le \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \le a(x)$$

for almost all $x \in \Omega$ *and* $2G(x,t) \le (\beta_0 - \lambda_k)t^2, \forall x \in \Omega, |t| \le r_0$*, where* $r_0 > 0$ *and* $\beta_0 \in (\lambda_{m-1}, \lambda_m)$*;*

(B_2)

$$2G(x,t) \ge (\lambda_{m-1} - \lambda_k)|t|^2$$

for all $x \in \Omega, t \in R$*;*
Then for almost all $\lambda \in \left(\frac{3\beta_0+\lambda_m}{2(\beta_0+\lambda_m)}, 1\right]$*, the problem (11) has two nontrivial solutions. Particularly, the problem (11) has infinitely many solutions. If, in addition,*

(B_3) *there exists* $\theta \in (1,2)$ *such that*

$$\liminf_{|t|\to\infty} \frac{tg(x,t) - 2G(x,t)}{|t|^\theta} > 0$$

uniformly for almost all $x \in \Omega$*. Then the problem (10) has two nontrivial solutions.*

Lemma 5 $G_\lambda(u) \to -\infty$ *uniformly for* $\lambda \in (0,1]$ *as* $\|u\| \to \infty, u \in N_m$.

Proof By the assumption (B_1), we can obtain that there exists $M > 0$ such that

$$2G(x,t) \ge (\lambda_m - \varepsilon)|t|^2, \ \ \forall \varepsilon > 0, \ \ |t| \ge M.$$

Hence,

$$
\begin{aligned}
G_\lambda(u) &= \frac{\lambda}{2}\|u\|^2 - \frac{\lambda_k}{2}\int_\Omega |u|^2dx - \int_\Omega G(x,u)dx \\
&\le \frac{\lambda_m}{2}\int_\Omega |u|^2dx - \frac{\lambda_k}{2}\int_\Omega |u|^2dx - \int_\Omega G(x,u)dx \\
&= \left[\frac{\lambda_m-\lambda_k}{2}\right]\int_\Omega |u|^2dx - \int_{\{|u|\ge M\}} G(x,u)dx - \int_{\{|u|<M\}} G(x,u)dx \\
&\le \left[\frac{\varepsilon-\lambda_k}{2}\right]\int_\Omega |u|^2dx + \left[\frac{\lambda_m-\varepsilon}{2}\right]\int_{\{|u|<M\}} |u|^2dx - \int_{\{|u|<M\}} G(x,u)dx \\
&\le \left[\frac{\varepsilon-\lambda_k}{2}\right]\int_\Omega |u|^2dx + c.
\end{aligned}
$$

It follows from the arbitrary property of ε, we can easily obtain the conclusion.

Lemma 6 *Under the assumptions of Theorem 5, there exist* $\rho_0 > 0$, $c_0 > 0$ *such that*

$$G_\lambda(u) \ge c_0 \quad \textit{for} \quad \|u\| = \rho_0,\ u \in N_{m-1}^\perp,$$

where $\lambda \in \left(\frac{3\beta_0+\lambda_m}{2(\beta_0+\lambda_m)}, 1\right]$.

Proof By the assumption (B_1), we see that there exists $r_1 > \lambda_m$ such that

$$2G(x,t) \le r_1 t^2, \quad \text{for } |t| \ge r_0,\ x \in \Omega, \tag{12}$$

where $r_1 > a(x) \ge \lambda_m$, for all $x \in \Omega$. Take $r_2 = 4r_1 - \lambda_k$. Then, from (12), by a simple computation, we have that

$$2G(x,t) \le r_2 t^2 - r_1 r_0^2, \quad \text{for } |t| \ge r_0,\ x \in \Omega. \tag{13}$$

For any $u \in N_{m-1}^\perp$, we write $u = v + w$ with $v \in E_m \bigoplus E_{m+1} \bigoplus \cdots \bigoplus E_{l-1}$ and $w \in N_{l-1}^\perp$, where l is large enough so that $\lambda_l > \frac{32\beta_0^2}{\lambda_m-\beta_0} + \frac{(48r_1+4\lambda_k)(\lambda_m+\beta_0)}{\lambda_m+3\beta_0}$. Let

$$\xi_1 = \frac{(r_2+\lambda_l)\lambda - 2\lambda_k}{4}w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\lambda_k}{4}v^2 - G(x, v+w). \tag{14}$$

If $|v + w| \le r_0$, then by the assumption (B_1) and the choice of λ_l, we have that

$$
\begin{aligned}
\xi_1 &\ge \frac{(r_2+\lambda_l)\lambda - 2\lambda_k}{4}w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\lambda_k}{4}v^2 - \frac{1}{2}(\beta_0-\lambda_k)(v+w)^2 \\
&\ge \frac{(r_2+\lambda_l)\lambda - 2\beta_0}{4}w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\beta_0}{4}v^2 - \frac{1}{2}(\beta_0-\lambda_k)|v||w| \\
&\ge \left[\frac{1}{2}([(r_2+\lambda_l)\lambda - 2\beta_0][(\beta_0+\lambda_m)\lambda - 2\beta_0])^{\frac{1}{2}} - (\beta_0-\lambda_k)\right]|v||w| \\
&\ge 0.
\end{aligned}
$$

If $|v + w| \geq r_0$, by (13), we have that

$$\begin{aligned}
\xi_1 &\geq \frac{(r_2+\lambda_l)\lambda - 2\lambda_k}{4} w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\lambda_k}{4} v^2 - \frac{1}{2}[r_2(v+w)^2 - r_1 r_0^2] \\
&= \frac{(r_2+\lambda_l)\lambda - 2\lambda_k - 2r_2}{4} w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\lambda_k - 2r_2}{4} v^2 - r_2 vw + \frac{r_1 r_0^2}{2} \\
&= \frac{(r_2+\lambda_l)\lambda - 2\lambda_k - 2r_2}{4} w^2 + \frac{(\beta_0+\lambda_m)\lambda - 2\lambda_k - 2r_2}{4} v^2 - r_2 vw + \frac{r_1 r_0^2}{2} \\
&= \frac{(r_2+\lambda_l)\lambda - 8r_1}{4} w^2 + \frac{(\beta_0+\lambda_m)\lambda - 8r_1}{4} v^2 - r_2 vw + \frac{r_1 r_0^2}{2} \\
&= \xi_2 + \xi_3,
\end{aligned}$$

where

$$\xi_2 = \frac{(r_2+\lambda_l)\lambda - 8r_1}{8} w^2 + \frac{(\lambda_m - \beta_0)\lambda}{4} v^2 - \lambda\beta_0 vw,$$

$$\xi_3 = \frac{(r_2+\lambda_l)\lambda - 8r_1}{8} w^2 - \frac{4r_1 - \lambda\beta_0}{2} v^2 - (r_2 - \lambda\beta_0) vw + \frac{r_1 r_0^2}{2}.$$

If

$$\frac{\lambda(\lambda_m - \beta_0)}{4} |v| - \lambda\beta_0 |w| \geq 0,$$

then

$$\begin{aligned}
\xi_2 &\geq \frac{(r_2+\lambda_l)\lambda - 8r_1}{8} w^2 + \left(\frac{\lambda(\lambda_m - \beta_0)}{4} |v| - \lambda\beta_0 |w| \right) |v| \\
&\geq 0.
\end{aligned}$$

If

$$\frac{\lambda(\lambda_m - \beta_0)}{4} |v| - \lambda\beta_0 |w| \leq 0,$$

by the choice of λ and λ_l, we have that

$$\frac{(r_2+\lambda_l)\lambda - 8r_1}{8} > \frac{4\lambda\beta_0^2}{\lambda_m - \beta_0}.$$

Thus,

$$\begin{aligned}
\xi_2 &\geq \frac{(r_2+\lambda_l)\lambda - 8r_1}{8} w^2 + \frac{\lambda(\lambda_m - \beta_0)}{4} v^2 - \lambda\beta_0 |w||v| \\
&\geq \left[\frac{(r_2+\lambda_l)\lambda - 8r_1}{8} - \frac{4\lambda\beta_0^2}{\lambda_m - \beta_0} \right] w^2 + \frac{\lambda(\lambda_m - \beta_0)}{4} v^2 \\
&\geq 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\xi_3 &\geq \frac{(r_2+\lambda_l)\lambda - 8r_1}{8}w^2 - \frac{4r_1 - \lambda\beta_0}{2}v^2 - (r_2 - \lambda\beta_0)|w||v| + \frac{r_1 r_0^2}{2}\\ &\geq \frac{(r_2+\lambda_l)\lambda - 8r_1}{8}w^2 - \frac{4r_1 - \lambda\beta_0}{2}v^2 - \frac{r_2 - \lambda\beta_0}{2}w^2 - \frac{r_2 - \lambda\beta_0}{2}v^2 + \frac{r_1 r_0^2}{2}\\ &= \frac{(r_2+\lambda_l)\lambda - 24r_1 + 4\lambda\beta_0 + 4\lambda_k}{8}w^2 - \frac{8r_1 - \lambda_k - 2\lambda\beta_0}{2}v^2 + \frac{r_1 r_0^2}{2}\\ &\geq -\frac{8r_1 - \lambda_k - 2\lambda\beta_0}{2}v^2 + \frac{r_1 r_0^2}{2}.\end{aligned}$$

Since $\dim N_{l-1} < \infty$, we may find a constant C_{l-1} such that

$$\max_{\Omega}|v| \leq C_{l-1}\|v\| \quad \text{for all } v \in N_{l-1}.$$

Let

$$\delta_0 = \frac{\beta_0}{2(\lambda_m+\beta_0)(8r_1 - \lambda_k - \frac{1}{2}\beta_0)C_{l-1}^2}\left(1 - \frac{\beta_0}{\lambda_m}\right)$$

then $\delta_0 > 0$. Let

$$\Omega_1 = \{x \in \Omega : |v+w| \leq r_0\}, \quad \Omega_2 = \{x \in \Omega : |v+w| \geq r_0\}.$$

Combing the above estimates, we have that

$$\begin{aligned}\int_\Omega \xi_1 dx &= \int_{\Omega_1} \xi_1 dx + \int_{\Omega_2} \xi_1 dx\\ &\geq \int_{\Omega_2} \xi_1 dx\\ &\geq \int_{\Omega_2} \xi_3 dx\\ &\geq -\frac{8r_1 - \lambda_k - 2\lambda\beta_0}{2}\int_{\Omega_2} v^2 dx + \frac{r_1 r_0^2}{2}\text{meas}\Omega_2.\end{aligned}$$

If meas $\Omega_2 \geq \delta_0$, then

$$\int_\Omega \xi_1 dx \geq -\frac{8r_1 - \lambda_k - 2\lambda\beta_0}{2\lambda_m}\|v\|^2 + \frac{r_1 r_0^2}{2}\delta_0. \tag{15}$$

If meas $\Omega_2 < \delta_0$, then

$$\begin{aligned}\int_\Omega \xi_1 dx &\geq -\frac{8r_1 - \lambda_k - 2\lambda\beta_0}{2}C_{l-1}^2\|v\|^2\text{meas}\Omega_2\\ &\geq -\frac{\beta_0}{4(\lambda_m+\beta_0)}\left(1 - \frac{\beta_0}{\lambda_m}\right)\|v\|^2. \end{aligned} \tag{16}$$

Hence, by (14) and the choice of λ, we have that

$$\begin{aligned}
G_\lambda(u) &= G_\lambda(v+w)\\
&= \frac{\lambda}{2}(\|v\|^2+\|w\|^2) - \frac{\lambda_k}{2}(\|v\|_2^2+\|w\|_2^2) - \int_\Omega G(x, v+w)dx\\
&\geq \frac{\lambda}{4}(\|v\|^2+\|w\|^2) + \frac{\lambda}{4}\lambda_m\|v\|_2^2 + \frac{\lambda}{4}\lambda_l\|w\|_2^2 - \frac{\lambda_k}{2}(\|v\|_2^2+\|w\|_2^2)\\
&\quad - \int_\Omega G(x, v+w)dx\\
&= \frac{\lambda}{4}(\|w\|^2 - r_2\|w\|_2^2) + \frac{\lambda}{4}\left(\|v\|^2 - \beta_0\|v\|_2^2\right) + \int_\Omega \xi_1 dx\\
&\geq \frac{\lambda}{4}\left(1-\frac{r_2}{\lambda_l}\right)\|w\|^2 + \frac{\lambda}{4}\left(1-\frac{\beta_0}{\lambda_m}\right)\|v\|^2 + \int_\Omega \xi_1 dx\\
&\geq \frac{\beta_0}{2(\lambda_m+\beta_0)}\min\left\{\left(1-\frac{r_2}{\lambda_l}\right), \left(1-\frac{\beta_0}{\lambda_m}\right)\right\}\|u\|^2 + \int_\Omega \xi_1 dx\\
&\geq \frac{\beta_0}{2(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|u\|^2 + \int_\Omega \xi_1 dx.
\end{aligned}$$

If meas$\Omega_2 \geq \delta_0$, by (15), then

$$\begin{aligned}
G_\lambda(u) &\geq \frac{\beta_0}{2(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|u\|^2 - \frac{8r_1-\lambda_k-2\lambda\beta_0}{2\lambda_m}\|v\|^2 + \frac{r_1 r_0^2}{2}\delta_0\\
&\geq \frac{\beta_0}{2(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|u\|^2 - \frac{8r_1-\lambda_k-2\lambda\beta_0}{2\lambda_m}\|u\|^2 + \frac{r_1 r_0^2}{2}\delta_0.
\end{aligned}$$

If meas$\Omega_2 < \delta_0$, by (16), then

$$\begin{aligned}
G_\lambda(u) &\geq \frac{\beta_0}{2(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|u\|^2 - \frac{\beta_0}{4(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|v\|^2\\
&\geq \frac{\beta_0}{4(\lambda_m+\beta_0)}\left(1-\frac{\beta_0}{\lambda_m}\right)\|u\|^2.
\end{aligned}$$

Consequently, we may find $\rho_0 > 0, c_0 > 0$ such that $G_\lambda(u) \geq c_0$ for $\|u\| = \rho_0, u \in N_{m-1}^\perp$, and $\lambda \in \left(\frac{3\beta_0+\lambda_m}{2(\beta_0+\lambda_m)}, 1\right]$.

Combining Lemma 5 and 6, with the similar proof as Theorem 3, we can complete Theorem 5.

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